Extremal black hole formation as a critical phenomenon

Ryan Unger

Stanford University & UC Berkeley

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joint work with Christoph Kehle (MIT)

Outline

1. Lightning review of general relativity and gravitational collapse

2. Critical phenomena in gravitational collapse

3. A new phenomenon: extremal critical collapse (and a bit of the construction)

4. The conjectured structure of moduli space near extremality

General relativity

Our setting today is EINSTEIN's theory of **general relativity**. A **spacetime** consists of a 4-manifold $\mathcal{M}^{3+1}$ and a Lorentzian metric g satisfying the **Einstein field equations**

$$Ric(g) - \frac{1}{2} R(g) g = 2T,$$

where T is the **energy-momentum tensor** of matter (scalar field, perfect fluid, kinetic model, ...).

Example: **Minkowski space.** $\mathcal{M}^{3+1} = \mathbb{R}^{3+1}_{t,x,y,z}$ and

$$g = -dt^2 + dx^2 + dy^2 + dz^2$$

This metric describes the geometry of special relativity with speed of light $c = 1$.

Given a vector $v \in T_p\mathcal{M}$:

- $g(v, v) < 0$, v is *timelike*
- $g(v, v) = 0$, v is *null*
- $g(v, v) > 0$, v is *spacelike*

Curves with null or timelike tangent vector let us define **causality** on a spacetime.

General relativity is a dynamical theory

$$Ric(g) - \frac{1}{2}R(g)g = 2T(\psi)$$

In appropriate coordinates, the Einstein equations constitute a system of nonlinear wave equations for the spacetime metric g and matter fields ψ:

$$g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + \mathcal{N}(g, \partial g) = \text{terms involving } \psi$$

$$\text{\& evolution equation for } \psi$$

Theorem (Choquet-Bruhat–Geroch '52, '69).
Any Cauchy data set $(\Sigma^3, \bar{g}, \bar{k}, \bar{\psi})$ for the Einstein equations coupled to a suitable matter model extends to a unique maximal "globally hyperbolic" development:

$$(\Sigma^3, \bar{g}, \bar{k}, \bar{\psi}) \overset{!}{\hookrightarrow} (\mathcal{M}^{3+1}, g, \psi).$$

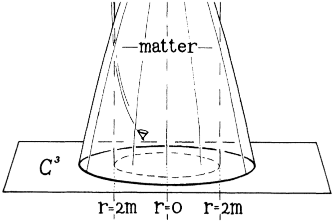
Given the state of the universe (the gravitational field and matter fields) at one instant of time, Einstein's equations uniquely determine its evolution for all later times.

Example: Minkowski space is the unique evolution of $(\mathbb{R}^3, \delta, 0, 0)$, which is geodesically complete and dispersive.

In this talk, we consider *asymptotically flat* data on $\Sigma^3 \cong \mathbb{R}^3$.

Death of a star in general relativity

LEMAÎTRE '33, OPPENHEIMER–SNYDER '39, PENROSE '65, solution of the Einstein–Euler equations:



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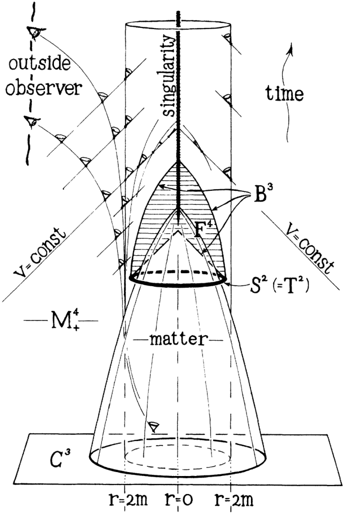

Death of a star in general relativity: gravitational collapse

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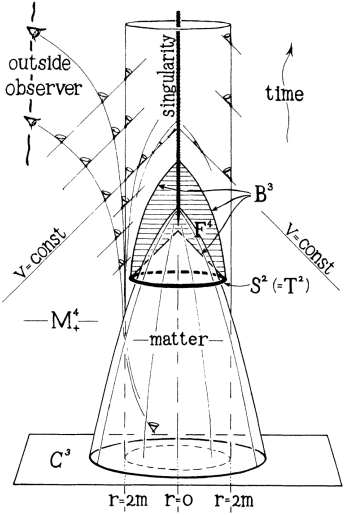

A **black hole** is a region of spacetime which cannot be seen by "far away" observers

Death of a star in general relativity: gravitational collapse

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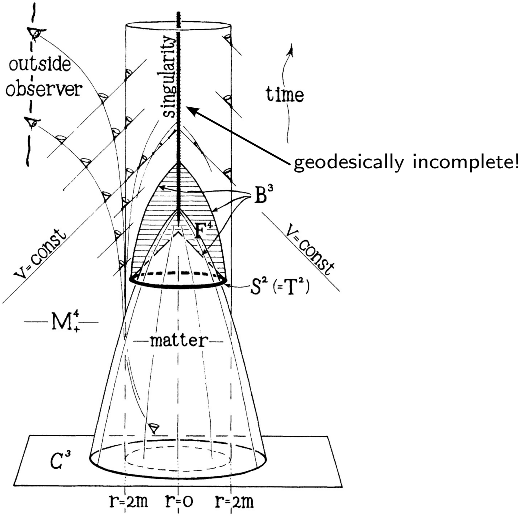
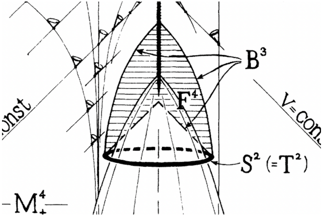


A **black hole** is a region of spacetime which cannot be seen by "far away" observers



Definition (Penrose '65).

A spacelike surface $\Sigma^2 \subset \mathcal{M}^4$ is *trapped* if its inner and outer null mean curvatures (null expansions) are negative.

$\Longleftrightarrow$ Any deformation of Σ towards the future strictly decreases its area.

Penrose's incompleteness theorem.

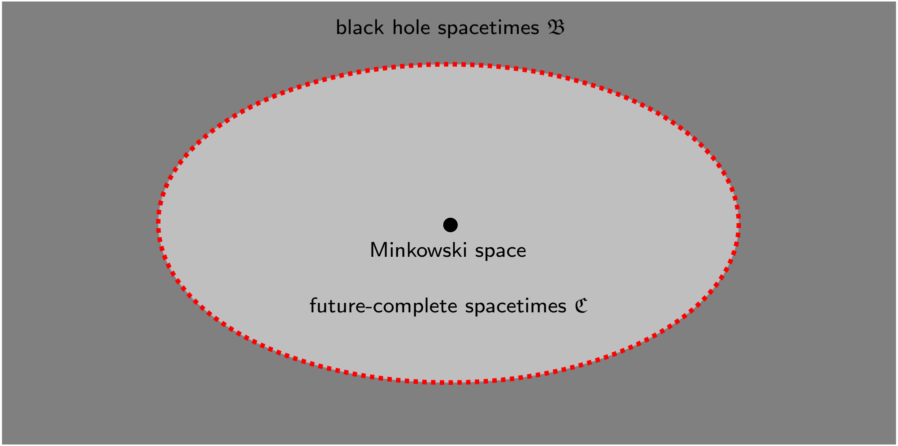
If an asymptotically flat spacetime contains a trapped surface and satisfies the null energy condition,

$$\mathrm{Ric}(X,X) \geq 0 \quad \forall X \text{ null} \iff T(X,X) \geq 0 \quad \forall X \text{ null},$$

then it is geodesically incomplete.

The property of having a trapped surface is stable, so this incompleteness is stable!

Conjectural cartoon picture of moduli space



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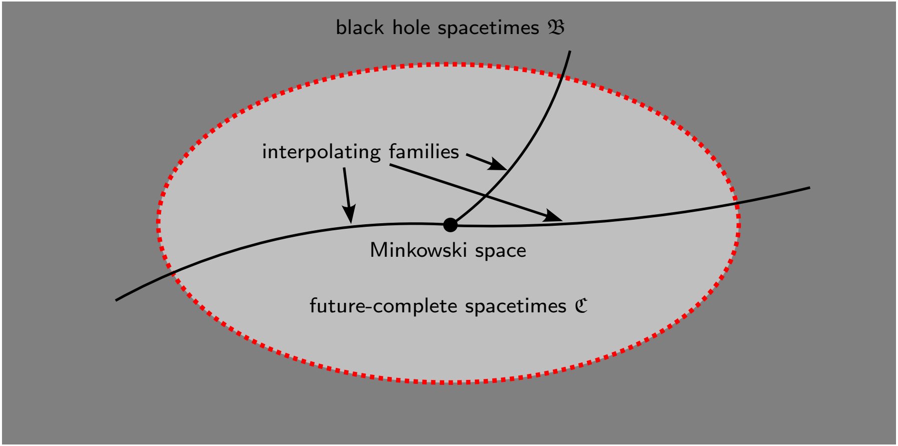
Critical collapse: study of the black hole formation threshold in moduli space ($\partial\mathfrak{C}/\partial\mathfrak{B}$)



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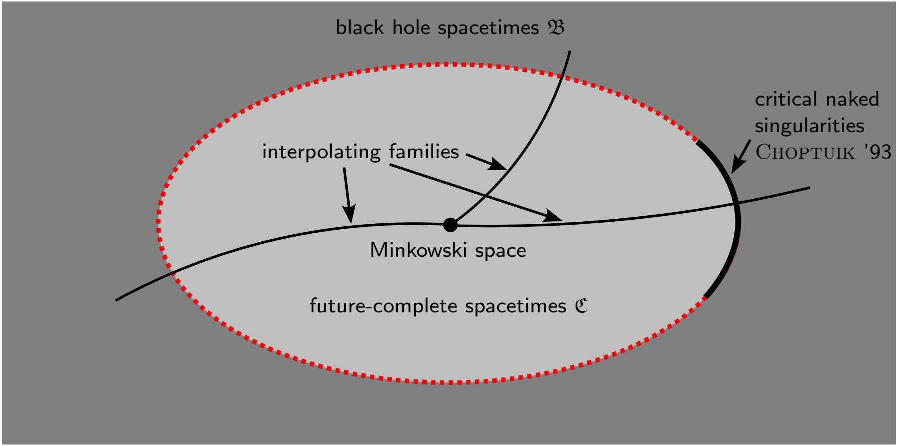

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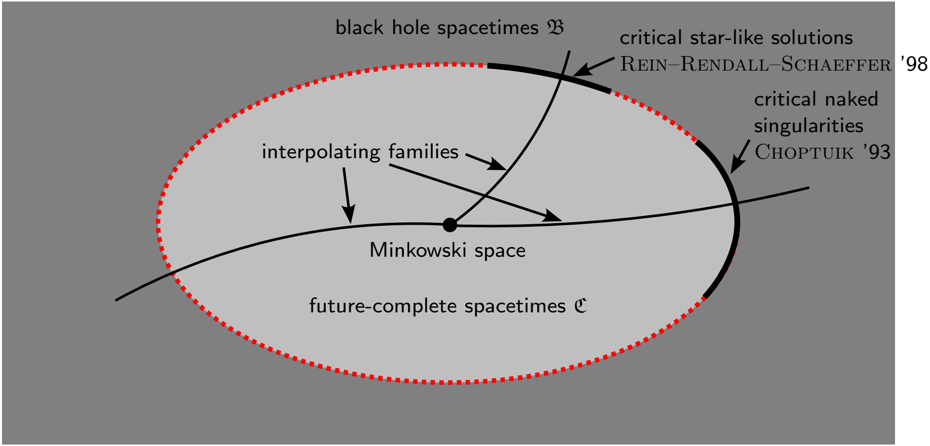

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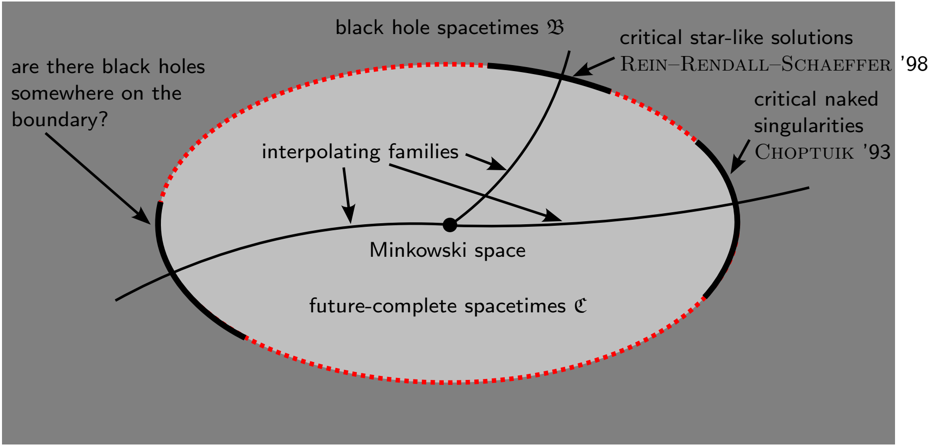


Theorem (Kehle–U. '24).
There exist black hole spacetimes on the black hole formation threshold in the Einstein–Maxwell–charged Vlasov model (self-gravitating charged collisionless plasma).

Moreover, these black holes are extremal.

This is the first rigorous construction of a critical solution in general relativity.
Curiously, these types of solutions were completely missed in numerical investigations.

- What are extremal black holes?
- What is the Einstein–Maxwell–charged Vlasov model?

Extremal black holes

- Standard stationary BH solutions characterized by mass M, charge e, specific angular momentum a
- **Extremal** BHs have maximal (e, a) given M

The **Schwarzschild solution** for mass $M > 0$:

$$g_M = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 g_{S^2}$$

- Solves the **Einstein vacuum** equations
- Describes a static, nonrotating black hole

Extremal black holes

- Standard stationary BH solutions characterized by mass M, charge e, specific angular momentum a
- **Extremal** BHs have maximal (e, a) given M

The **Reissner–Nordström solution** for mass $M > 0$ and charge $e \in \mathbb{R}$:

$$g_{M,e} = -\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2 g_{S^2}$$

- Solves the **Einstein–Maxwell** equations
- Describes a static, nonrotating, charged black hole if $|e| \leq M$
  - $|e| < M$ **subextremal** (includes Schwarzschild!)
  - $|e| = M$ **extremal**
  - $|e| > M$ **superextremal** (not a black hole!)

Extremal black holes and trapped surfaces

Fundamental fact.
The interior of a subextremal stationary black hole is foliated by trapped surfaces. Extremal stationary black holes do not contain trapped surfaces, but are nevertheless geodesically incomplete.

Penrose's incompleteness theorem.
Trapped surfaces $+$ *Einstein's equations* $\Longrightarrow$ *geodesically incomplete.*

Corollary.
Subextremal black holes cannot model critical behavior in gravitational collapse.

Extremality is unavoidable in our theorem.

Aside: the third law of black hole thermodynamics

- NERNST's theorem in classical thermodynamics (1912): *The temperature T of a body cannot reach absolute zero in a finite process.*
- BARDEEN, CARTER, and HAWKING proposed in 1973 a fundamental analogy between black hole mechanics and classical thermodynamics.
- Area of the event horizon $A \iff$ entropy S
- Surface gravity $\kappa \iff$ temperature T

The Third Law

It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

$$\kappa(g_{M,e}) = \frac{\sqrt{M^2 - e^2}}{(M + \sqrt{M^2 - e^2})^2}$$

The third law would imply that extremal black holes cannot form dynamically.

Theorem (Kehle–U. '22).
Extremal black holes can form dynamically in the gravitational collapse of spherically symmetric charged scalar field. The third law of black hole thermodynamics is false.

The Einstein–Maxwell–charged Vlasov system

- Spacetime $(\mathcal{M}^4, g)$, electromagnetic field $F \in \Omega^2$, $dF = 0$
- Distribution function $f(x,p) \geq 0$ defined on
$$P^{\mathfrak{m}} \doteq \{(x,p) \in T\mathcal{M} : g_x(p,p) = -\mathfrak{m}^2, p \text{ future directed}\},$$
models a collisionless gas of massive or massless particles with charge $\mathfrak{e}$
- GR version of the ubiquitous SR Vlasov–Maxwell model
- Einstein field equations:
$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = 2 \underbrace{\left(F_{\mu}{}^{\alpha} F_{\nu\alpha} - \tfrac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right)}_{T^{\mathrm{EM}}_{\mu\nu}} + 2 \underbrace{\int_{P^{\mathfrak{m}}_x} p_\mu p_\nu f \, d\mu}_{T^{\mathrm{Vlasov}}_{\mu\nu}}$$
- Maxwell's equations:
$$\nabla^\alpha F_{\mu\alpha} = \mathfrak{e} \underbrace{\int_{P^{\mathfrak{m}}_x} p_\mu f \, d\mu}_{N_\mu}, \quad J^{\mathrm{EM}} = \mathfrak{e} N$$
- Vlasov equation:
$$\left(p^\mu \frac{\partial}{\partial x^\mu} - \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \frac{\partial}{\partial p^\mu} + \mathfrak{e} F^\mu{}_\alpha p^\alpha \frac{\partial}{\partial p^\mu} \right) f = 0$$
- f is constant along trajectories $\gamma : I \to \mathcal{M}$ of the *Lorentz force*
$$\dot\gamma^\nu \nabla_\nu \dot\gamma^\mu = \mathfrak{e} F^\mu{}_\nu \dot\gamma^\nu,$$
also known as **electromagnetic geodesics**

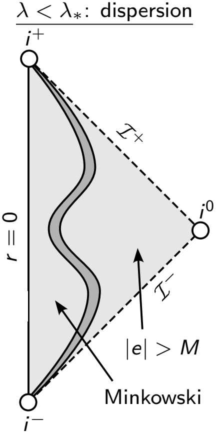
Precise statement of the theorem

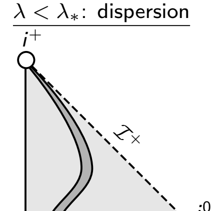
Theorem (Kehle–U. ’24).
There exist C^∞ 1-parameter families of spherically symmetric solutions $\{\mathcal{D}_\lambda\}_{\lambda\in[0,1]}$ to the EMV system with the following properties:

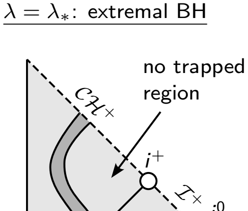
1. *$\mathcal{D}_0$ is Minkowski space.There exists $\lambda_* \in (0,1)$ such that for $\lambda < \lambda_*$, $\mathcal{D}_\lambda$ is geodesically complete and decays to Minkowski space.*
2. *If $\lambda = \lambda_*$, an* **extremal** *Reissner–Nordström black hole forms. The spacetime contains no trapped surfaces.*
3. *If $\lambda > \lambda_*$, a* **subextremal** *Reissner–Nordström black hole forms. The spacetime contains an open set of trapped surfaces.*

Moreover, for every $\lambda \in [0,1]$, $\mathcal{D}_\lambda$ is past causally geodesically complete.

Penrose diagrams of extremal critical collapse (massive particles)







Sketch of the proof: bouncing charged Vlasov beams

The problem is to construct charged Vlasov beams that:

- form a Reissner–Nordström exterior with specified parameters
- take advantage of EM repulsion and avoid getting too close to $r = 0$
- only form trapped surfaces where we want them to

Toy model: Ori's bouncing charged null dust

- Charged VAIDYA metric: "time-dependent" Reissner–Nordström metric

$$g = -\left(1 - \frac{2M(v)}{r} + \frac{e(v)^2}{r^2}\right) dv^2 + 2dvdr + r^2 g_{S^2}$$

- ORI ('91) realized the charged Vaidya metric is a solution of the **Einstein–Maxwell-charged null dust model** (charged, massless, pressureless perfect fluid with momentum k and density ρ)

$$\begin{aligned} g_{\mu\nu} k^\mu k^\nu &= 0 \\ R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} &= 2\left(T^{\mathrm{EM}}_{\mu\nu} + \rho k_\mu k_\nu\right) \\ \nabla^\alpha F_{\mu\alpha} &= \mathfrak{e}\rho k_\mu \\ k^\nu \nabla_\nu k^\mu &= \mathfrak{e} F^\mu{}_\nu k^\nu \\ \nabla_\mu(\rho k^\mu) &= 0, \end{aligned}$$

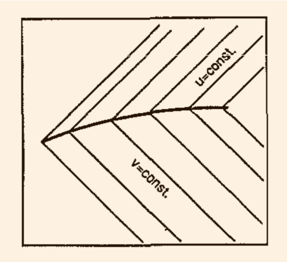
provided we set

$$k = \frac{\mathfrak{e}}{\dot{e}}\left(\dot{M} - \frac{e\dot{e}}{r}\right)(-\partial_r), \quad \rho = \frac{\dot{e}^2}{\mathfrak{e}^2 r^2}\left(\dot{M} - \frac{e\dot{e}}{r}\right)^{-1}$$

Toy model: Ori's bounce charged null dust

$$k^\mu = \frac{dx^\mu}{ds}, \quad \nabla_k k^\mu = \mathfrak{e} F^\mu{}_\nu k^\nu$$

If (g, F) is spherically symmetric and k is radial, then k is an eigenvector of F
$\implies$ $k(s)$ can decay exponentially, $x(s)$ has a limit point at $r = e\dot{e}/\dot{M}$ as $s \to +\infty$



Proposition (Kehle–U. '24).

There exists a procedure for generating bouncing charged null dust spacetimes where the location of the bounce and initial and final Reissner–Nordström parameters can be prescribed.

Singular toy model: Ori's bouncing charged null dust

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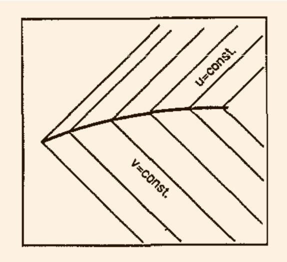

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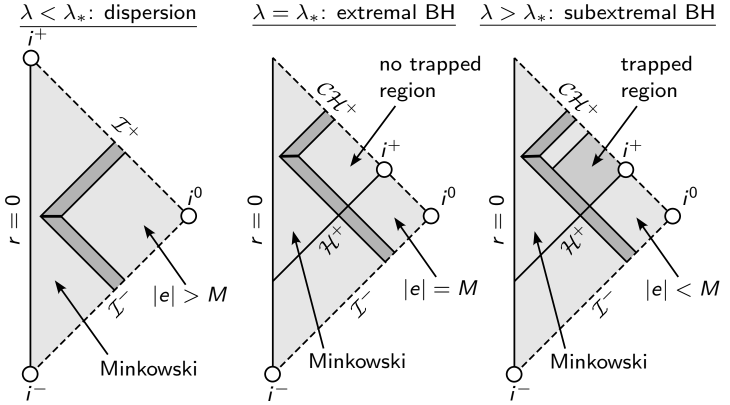
ECC in Ori's model

Theorem (Kehle–U. '24).
Ori's charged null dust model exhibits extremal critical collapse.



Upshot

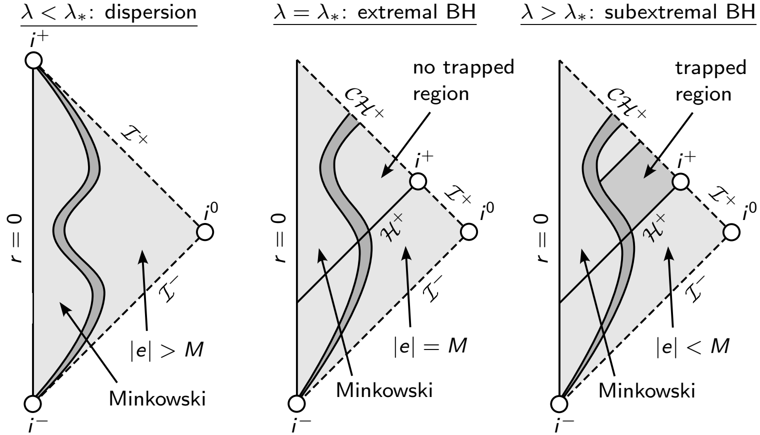
Studying this toy model does give valuable insight into constructing **smooth extremal critical collapse**!

Theorem (Kehle–U. '24).

Solutions of Ori's model can be realized as suitable limits of C^∞ solutions of the spherically symmetric Einstein–Maxwell–charged Vlasov model (C^1 for g and weak* for the matter).

In kinetic theory language, Ori's model arises as a hydrodynamic limit of Einstein–Maxwell–Vlasov.

Construction of bouncing charged Vlasov beams



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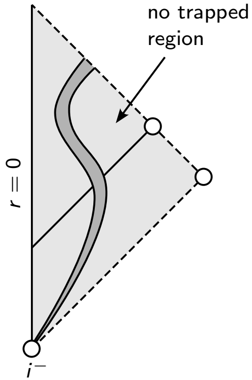

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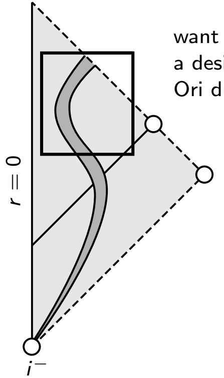
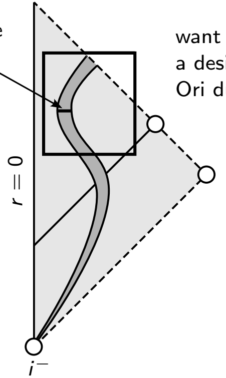


want this region to look like a desingularized bouncing Ori dust solution

Construction of bouncing charged Vlasov beams

approximate spacelike bounce hypersurface



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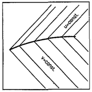

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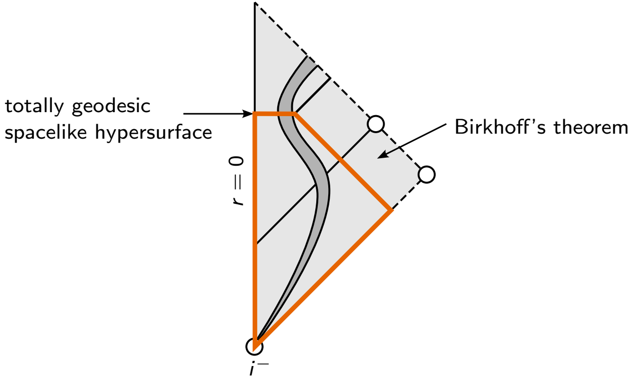

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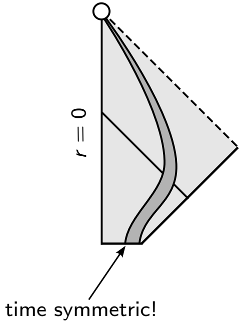


particles have conserved
angular momentum $\ell^2 = r^2 \not{g}(p, p)$

charged null dust:
monokinetic Maxwell–Vlasov with $\ell = 0$

bouncing charged null dust has $p = 0$
along the bounce hypersurface

initial data f_0 needs to have $p \approx \ell \approx \varepsilon \ll 1$
to behave like dust

$|J^{\mathrm{EM}}| \gtrsim 1 \Longrightarrow f_0 \approx \varepsilon^{-3}$ ($f \rightharpoonup \delta'(p)$ as $\varepsilon \to 0$)

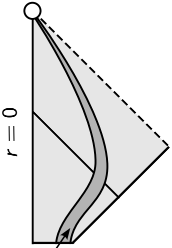
dust approximation requires a **singular ansatz** for f_0

f_0 is given by an explicit formula

Construction of bouncing charged Vlasov beams

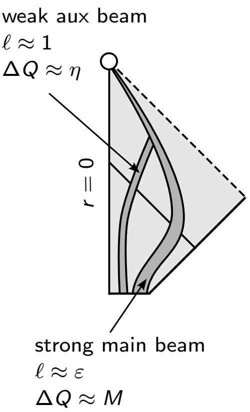
the most important feature to resolve is the
outward acceleration near the bounce hypersurface

we employ a weak "auxiliary beam" to impart charge
$0 < \varepsilon \ll \eta \ll 1 \Longrightarrow$ **stabilizes** the main beam



strong main beam
$\ell \approx \varepsilon$
$\Delta Q \approx M$

Construction of bouncing charged Vlasov beams



the most important feature to resolve is the outward acceleration near the bounce hypersurface

we employ a weak "auxiliary beam" to impart charge $0 < \varepsilon \ll \eta \ll 1 \implies$ **stabilizes** the main beam

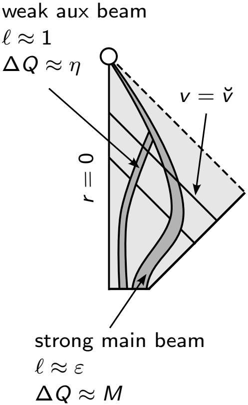
the auxiliary beam bounces due to angular momentum repulsion, not charge

null structure: T^{uu}, T^{uv} better in ε than T^{vv}

monotonicity: $\partial_u Q \leq 0$, $\partial_v Q \geq 0$

dispersion proved using energy estimates at a late time $\check{v} \gg 1$

Construction of bouncing charged Vlasov beams



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hierarchy of scales $0 < \mathfrak{m} \ll \varepsilon \ll \eta \ll \check{v}^{-1} \ll 1$

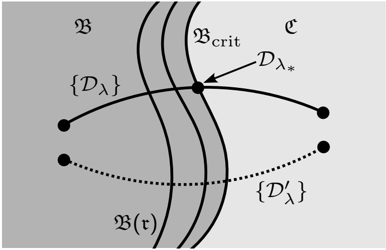
Conjectured stability of extremal critical collapse

Conjecture.

Extremal critical collapse is stable:

Let $\{\mathcal{D}_\lambda\}$ be one of the interpolating families given by [KU24]. Then there exists a C^1 hypersurface $\mathfrak{B}_{\mathrm{crit}}$ of the spherically symmetric moduli space $\mathfrak{M}$ such that $\mathcal{D}_{\lambda_*} \in \mathfrak{B}_{\mathrm{crit}} \subset \mathfrak{B}$, which has the following properties:

1. *$\mathfrak{B}_{\mathrm{crit}}$ is critical in the sense that $\mathfrak{B}$ and $\mathfrak{C}$ locally lie on opposite sides of $\mathfrak{B}_{\mathrm{crit}}$.*
2. *If $\mathcal{D} \in \mathfrak{B}_{\mathrm{crit}}$, then $\mathcal{D}$ contains a black hole which asymptotically settles down to a extremal Reissner–Nordström.*



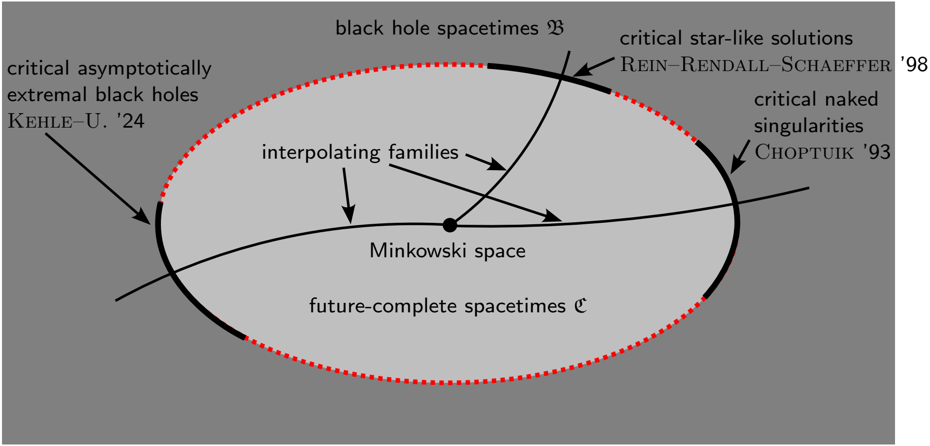
- This is also a nontrivial statement about **interiors** of extremal black holes
- Fundamental difficulty: Aretakis instability associated to extremal horizons

The vacuum case: extremal critical collapse

In principle, *extremal critical collapse*, its *stability*, and the *conjectured picture of moduli space* can be conjectured to also hold true in **vacuum** with extremal Reissner–Nordström replaced by **extremal Kerr**.

However, this is a very difficult problem which also relates to understanding

- the formation of extremal black holes in vacuum (the case $|a| \ll M$ has been resolved in Kehle–U. '23)
- the stability and codimension stability of subextremal and extremal black holes in vacuum and without symmetry assumptions
- the nonlinear ramifications of horizon instabilities associated to extremal Kerr, in particular the newly discovered **azimuthal instabilities** (Gajic '23) which are **stronger** than the Aretakis instability


black hole spacetimes $\mathfrak{B}$
critical star-like solutions
REIN–RENDALL–SCHAEFFER '98
critical asymptotically
extremal black holes
KEHLE–U. '24
critical naked
singularities
CHOPTUIK '93
interpolating families
Minkowski space
future-complete spacetimes $\mathfrak{C}$

Thank you!